

ON NON-CONTRACTIBLE PERIODIC ORBITS OF HAMILTONIAN DIFFEOMORPHISMS

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ABSTRACT. We prove that any Hamiltonian diffeomorphism of a closed symplectic manifold equipped with an atoroidal symplectic form has simple non-contractible periodic orbits of arbitrarily large period, provided that the diffeomorphism has a non-degenerate (or even isolated and homologically non-trivial) periodic orbit with non-zero homology class and the set of one-periodic orbits in that class is finite.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the question of existence of non-contractible periodic orbits for Hamiltonian diffeomorphisms φ_H of a closed symplectic manifold (M^{2n}, ω) such that the integral of ω over all toroidal classes in $H_2(M; \mathbb{Z})$ vanishes. Under this condition, we prove that φ_H has non-contractible periodic orbits of arbitrarily large prime period, provided that it has one non-degenerate (or isolated and homologically non-trivial) periodic orbit with homology class non-zero modulo torsion.

Non-contractible periodic orbits of Hamiltonian systems have been previously investigated in a number of papers using Floer theoretical techniques; see, e.g., [BPS, BuHal, GaL, Lee, Ni, We]. The central theme of this paper, however, lies in a different domain. Conceptually, the key difference is that here we allow the period of the orbits to vary, whereas the above works concern the existence of periodic orbits (in a given free homotopy class) of a fixed period. In other words, we approach the problem from a dynamical systems perspective rather than topological. On the technical side, the source of periodic orbits in those works is non-vanishing Floer homology or other more subtle invariants such as Floer-theoretic torsion. In general,

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one major obstacle to employing this method in the investigation of non-contractible periodic orbits for closed manifolds is that the total non-contractible Floer homology is zero since all one-periodic orbits of a C^2 -small autonomous Hamiltonian are constant, and hence contractible. (Thus even the entire Floer complex may be zero.) Therefore, the previous results mainly apply to open manifolds such as ordinary and twisted cotangent bundles; see, e.g., [BPS, Ni, We]. Working with closed manifolds, we take a different approach. In our case, the source for periodic orbits is the change in certain filtered Floer homology groups under the iteration of the Hamiltonian, and the original non-contractible orbit spawns infinitely many other orbits. (Our argument shares many common elements with that [Gü].)

To put our result in perspective in a different way, recall the following variant of the Conley conjecture, referred to as the HZ-conjecture in [Gü]. This is the assertion that a Hamiltonian diffeomorphism with “more than necessary” non-degenerate (or just homologically non-trivial) fixed points has infinitely many periodic orbits. Here “more than necessary” is usually interpreted as a lower bound arising from some version of the Arnold conjecture. For \mathbb{CP}^n , the expected threshold is $n + 1$. This conjecture, originally stated in [HZ], is inspired by a celebrated theorem of Franks stating that a Hamiltonian diffeomorphism (or, even, an area preserving homeomorphism) of S^2 with at least three fixed points must have infinitely many periodic orbits, [Fr1, Fr2]; see also [FH, LeC] for further refinements and [BraHo, CKRTZ, Ker] for symplectic topological proofs.

Viewing the HZ-conjecture in a broader context is often useful. Namely, it appears that the presence of a fixed point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic orbits. For instance, a theorem from [GG4] asserts that, for a certain class of closed monotone symplectic manifolds including \mathbb{CP}^n , any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits; see also [Gü] for other results along these lines. (Note also that these considerations are related to and motivated by the circle of questions in the realm of the original Conley conjecture; see, [FH, Gi2, GG3, Hi1, LeC, SZ] and also [CGG, GG1, He] for a detailed account of this conjecture and related results.)

The main result of the paper therefore can be viewed as the proof of the generalized HZ-conjecture for non-contractible periodic orbits of Hamiltonian diffeomorphisms. Indeed, observe that from a homological perspective a Hamiltonian diffeomorphism of a closed symplectic manifold need not have a non-contractible periodic orbit. However, our result shows that, if such an orbit exists, it generates infinitely many others.

Let us now state the main result of the paper. Let (M^{2n}, ω) be a closed symplectic manifold equipped with an *atoroidal* symplectic form ω . (Namely, we assume that for every map $v: \mathbb{T}^2 \rightarrow M$, the integral of ω over v vanishes, i.e., $\langle [\omega], [v] \rangle = 0$.) For a Hamiltonian diffeomorphism φ_H of M , denote by $\mathcal{P}_1(\varphi_H, [\gamma])$ the collection of one-periodic orbits of φ_H with homology class $[\gamma] \in H_1(M; \mathbb{Z})/\text{Tor}$.

Theorem 1.1. *Let M be a closed symplectic manifold equipped with an atoroidal symplectic form ω . Assume that a Hamiltonian diffeomorphism φ_H of M has a non-degenerate one-periodic orbit γ with homology class $[\gamma] \neq 0$ in $H_1(M; \mathbb{Z})/\text{Tor}$ and that $\mathcal{P}_1(\varphi_H, [\gamma])$ is finite. Then, for every sufficiently large prime p , the diffeomorphism φ_H has a simple periodic orbit with homology class $p[\gamma]$ and period either p or p' , where p' is the first prime number greater than p .*

Corollary 1.2. *In the setting of Theorem 1.1 (or Theorem 3.1), the number of non-contractible periodic orbits with period less than or equal to k , or the number of distinct homology classes represented by such orbits, is bounded from below by $\text{const} \cdot k / \log k$.*

An immediate consequence of Theorem 1.1 is that φ_H has infinitely many periodic orbits with homology classes in $\mathbb{N}[\gamma]$ whether or not $\mathcal{P}_1(\varphi_H, [\gamma])$ is finite. Moreover, the non-degeneracy condition in the theorem can be relaxed and replaced by a much weaker, albeit more technical, requirement that γ is isolated and *homologically non-trivial*, i.e., its local Floer homology is non-zero. (See Section 3 for a discussion of this notion.) Finally, the orbit γ need not be one-periodic; the theorem (with obvious modifications) still holds when γ is just a periodic orbit.

Among the manifolds meeting the requirements of the theorem are, for instance, all closed Kähler manifolds with negative sectional curvature (e.g., complex hyperbolic spaces) or, more generally, any closed symplectic manifold M such that $\pi_2(M) = 0$ and $\pi_1(M)$ is a hyperbolic group. Indeed, in this case any map $v: \mathbb{T}^2 \rightarrow M$ is homologous to zero over \mathbb{R} , since $\pi_1(M)$ contains no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (see, e.g., [BriHae, GhdIH]), and hence the pull-back homomorphism

$$v^*: H^2(M; \mathbb{R}) = H^2(K(\pi_1(M, 1); \mathbb{R})) \rightarrow H^2(\mathbb{T}^2; \mathbb{R})$$

is necessarily zero. (See [BruK, Ked] and also references therein for other examples.)

The existence of a non-contractible periodic orbit γ as in Theorem 1.1 appears to be a common occurrence. In fact, we are not aware of any example of a Hamiltonian diffeomorphism with isolated fixed points which obviously would not have a non-contractible periodic orbit, provided that $H_1(M; \mathbb{R}) \neq 0$. However, let us emphasize that the homology class of γ cannot be prescribed in advance, for, as simple examples (e.g., the height function on \mathbb{T}^2) show, non-contractible orbits need not exist in a given homology class or in the multiples of a given class. (In fact, KAM theory implies that this is even generically the case, at least in dimension two.)

The growth rate established in Corollary 1.2 is typical in this class of questions in the absence of hyperbolicity. For instance, a similar growth rate is known to hold in the context of the Conley conjecture (see, e.g., [Gi2, GG1, He, Hi1, SZ]) and for closed geodesics on S^2 ; see, e.g., [Hi2]. (There are, however, stronger growth results; see, e.g., [LeC, Vi].) Perhaps it is also worth pointing out that Theorem 1.1 and Corollary 1.2 are reminiscent of some results on the growth of prime geodesics on negatively curved manifolds (with the scale exponentially adjusted); see, e.g., [AS, Ma, PS]. However, to the best of our understanding, there is essentially no connection, technical or conceptual, between the respective results beyond a superficial resemblance.

Finally, the question of possible generalizations of Theorem 1.1 to the case when M is toroidally monotone will be addressed elsewhere. (Recall that M is said to be toroidally monotone if for every map $v: \mathbb{T}^2 \rightarrow M$ we have $\langle [\omega], [v] \rangle = \lambda \langle c_1(TM), [v] \rangle$ for some constant $\lambda \geq 0$ independent of v . See [GG4] for a partial result in this direction.)

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2. PRELIMINARIES

In this section, we set conventions and notation, and discuss Floer homology for non-contractible periodic orbits.

2.1. Conventions and notation. Throughout the paper, we assume that (M^{2n}, ω) is a closed manifold equipped with an *atoroidal* symplectic form ω , i.e., for every map $v: \mathbb{T}^2 \rightarrow M$, the integral of ω over v vanishes: $\langle [\omega], [v] \rangle = 0$. We refer the reader to Section 1 for examples of such (open or closed) symplectic manifolds and here only note that they arise naturally as the simplest setting where Floer theory for non-contractible periodic orbits is defined.

All Hamiltonians H are assumed to be one-periodic in time, i.e., $H: S^1 \times M \rightarrow \mathbb{R}$, and we set $H_t = H(t, \cdot)$ for $t \in S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. The (time-dependent) flow of X_H is denoted by φ_H^t and its time-one map by φ_H . Such time-one maps are referred to as *Hamiltonian diffeomorphisms*. The Hofer norm of H is $\|H\| = \int_{S^1} [\max_M H_t - \min_M H_t] dt$.

Let K and H be two one-periodic Hamiltonians. The composition $K \natural H$ is defined by the formula

$$(K \natural H)_t = K_t + H_t \circ (\varphi_K^t)^{-1},$$

and the flow of $K \natural H$ is $\varphi_K^t \circ \varphi_H^t$. We set $H^{\natural k} = H \natural \dots \natural H$ (k times). Abusing terminology, we will refer to $H^{\natural k}$ as the k th iteration of H . (Note that the flow $\varphi_{H^{\natural k}}^t = (\varphi_H^t)^k$, $t \in [0, 1]$, is homotopic with fixed end-points to the flow φ_H^t , $t \in [0, k]$.)

In general, $H^{\natural k}$ is not one-periodic, even when H is. However, this is the case if, for example, $H_0 \equiv 0 \equiv H_1$. The latter condition can be met by reparametrizing the Hamiltonian as a function of time without changing the time-one map; the Hofer norm, and actions and indices of the periodic orbits do not change during the procedure. Thus, in what follows, we always treat $H^{\natural k}$ as a one-periodic Hamiltonian.

The k th iteration of a one-periodic orbit γ of H is denoted by γ^k . More specifically, $\gamma^k(t) = \varphi_{H^{\natural k}}^t(\gamma(0))$, where $t \in [0, 1]$. We can think of γ^k as the k -periodic orbit $\gamma(t)$, $t \in [0, k]$, of H . Hence, there is an action-preserving one-to-one correspondence between one-periodic orbits of $H^{\natural k}$ and k -periodic orbits of H .

2.2. Floer homology for non-contractible periodic orbits. The key tool used in the proof of Theorem 1.1 is the filtered Floer homology for non-contractible periodic orbits of Hamiltonian diffeomorphisms. Various flavors of Floer homology in this case for both open and closed manifolds have been considered in several other works; see, e.g., [BPS, BuHal, GaL, GG4, Lee, Ni, We]. Below we briefly describe the elements of the construction relevant to our case.

Let ζ be a free homotopy class of maps $S^1 \rightarrow M$. Fix a *reference loop* $z \in \zeta$ and a symplectic trivialization of TM along z . (In fact, it would be sufficient to fix a trivialization of the canonical bundle of M along z .) A capping of $x: S^1 \rightarrow M$ with free homotopy class $\llbracket x \rrbracket = \zeta$ is a cylinder (i.e., a homotopy) $\Pi: [0, 1] \times S^1 \rightarrow M$ connecting x and z . The action of a one-periodic Hamiltonian H on x is

$$\mathcal{A}_H(x) = - \int_{\Pi} \omega + \int_{S^1} H_t(x(t)) dt.$$

The action $\mathcal{A}_H(x)$ is well-defined, i.e., independent of capping, for ω is atoroidal and hence $\langle [\omega], [v] \rangle = 0$, where $v: \mathbb{T}^2 \rightarrow M$ is the torus obtained by attaching two different capings to each other. Moreover, the critical points of \mathcal{A}_H are exactly

one-periodic orbits of the time-dependent Hamiltonian flow φ_H^t with homotopy class in ζ . The action spectrum $\mathcal{S}(H, \zeta)$ is the set of critical values of \mathcal{A}_H . It has zero measure; see, e.g., [HZ].

Observe that the trivialization of TM along the reference loop z uniquely extends to a trivialization along Π , and hence induces a trivialization along x . Using this trivialization, the *Conley–Zehnder index* $\mu_{\text{CZ}}(H, x) \in \mathbb{Z}$ of a *non-degenerate* orbit x is defined as in [Sa, SZ]. (Here x is said to be non-degenerate if the linearized return map $d\varphi_H: T_{x(0)}M \rightarrow T_{x(0)}M$ does not have one as an eigenvalue.) Similarly to the contractible case, the Conley–Zehnder index is defined only modulo $2N$, where N is the minimal “toroidal” Chern number of M , unless $c_1(TM)$ vanishes on toroidal homology classes in $H_2(M; \mathbb{Z})$. Note that, by definition, N is the positive generator of the subgroup of \mathbb{Z} generated by the integrals of $c_1(TM)$ over all toroidal classes.

The construction of Floer homology $\text{HF}(H, \zeta)$ and the filtered Floer homology $\text{HF}^{(a, b)}(H, \zeta)$, taking into account only the one-periodic orbits with homotopy classes in ζ , goes through exactly as in the contractible case; see, e.g., [BPS, Lee].

Recall that the total Floer homology in a class $\zeta \neq 0$ is trivial: $\text{HF}(H, \zeta) = 0$; for all one-periodic orbits of a C^2 -small autonomous Hamiltonian are constant, and hence contractible. However, the filtered Floer homology for some action interval is obviously non-zero whenever there exists an isolated and homologically non-trivial (e.g., non-degenerate) periodic orbit $\gamma \in \zeta$. Indeed, assuming for simplicity that $\mathcal{A}_H(\gamma) = 0$, observe that $\text{HF}^{(-a, a)}(H, \llbracket \gamma \rrbracket) \neq 0$, for instance, when $(-a, a)$ contains only the action value zero for $a \notin \mathcal{S}(H, \llbracket \gamma \rrbracket)$. This follows from the fact that

$$\text{HF}^{(-a, a)}(H, \llbracket \gamma \rrbracket) = \text{HF}(\gamma) \oplus \dots \neq 0,$$

where $\text{HF}(\gamma)$ is the local Floer homology of H at γ (see, e.g., [F11, F12, Gi2, GG2]), and the dots represent the local Floer homology contributions from other orbits in the class $\llbracket \gamma \rrbracket$ with action equal to zero; see, e.g., [GG2] for a proof of this fact.

The action and the index of periodic orbits (and hence the grading and filtration of the Floer complex) do depend on the choice of reference curves z and, in the index case, on the trivialization of TM along z . Whenever we consider the iteration $H^{\sharp \kappa}$ of H , we simultaneously iterate each class ζ and the reference curve z (i.e., we pass to ζ^κ and z^κ). As a consequence, the action functional on *iterated* κ -periodic orbits is homogeneous with respect to the iterations, i.e.,

$$\mathcal{A}_{H^{\sharp \kappa}}(x^\kappa) = \kappa \mathcal{A}_H(x). \quad (2.1)$$

3. PROOF OF THEOREM 1.1

As has been mentioned in the introduction, we establish a more general result. To state it, recall that an isolated periodic orbit γ of φ_H is said to be homologically non-trivial if the local Floer homology of H at γ is non-zero. For instance, a non-degenerate fixed point is homologically non-trivial. More generally, an isolated fixed point with non-vanishing topological index is homologically non-trivial; for this index is equal, up to a sign, to the Euler characteristic of the local Floer homology. The notion of homological non-triviality seems to be particularly well-suited for use in the context of Conley conjecture-type questions; we refer the reader to [Gü] for a detailed discussion of this condition. Theorem 1.1 is an immediate consequence of the following result.

Theorem 3.1. *Let M be a closed symplectic manifold equipped with an atoroidal symplectic form ω . Assume that a Hamiltonian diffeomorphism φ_H of M has an isolated and homologically non-trivial one-periodic orbit γ with homology class $[\gamma] \neq 0$ in $H_1(M; \mathbb{Z})/\text{Tor}$ and that $\mathcal{P}_1(\varphi_H, [\gamma])$ is finite. Then, for every sufficiently large prime p , the diffeomorphism φ_H has a simple periodic orbit with homology class $p[\gamma]$ and period either p or p' , where p' is the first prime number greater than p .*

Here, as in Section 1, $\mathcal{P}_1(\varphi_H, [\gamma])$ is the collection of one-periodic orbits of φ_H with homology class $[\gamma]$ in $H_1(M; \mathbb{Z})/\text{Tor}$.

Remark 3.2. In fact, the homology class $p[\gamma]$ in the assertion of the theorem can be replaced by the free homotopy class $\llbracket \gamma \rrbracket^p := \llbracket \gamma^p \rrbracket$, provided that $\llbracket \gamma \rrbracket \neq 0$ and, for instance, the only solutions to the equation $\llbracket \gamma \rrbracket^p = \zeta^q$, where p and q are sufficiently large primes and ζ is a free homotopy class, are $\zeta = \llbracket \gamma \rrbracket$ and $p = q$.

Proof of Theorem 3.1. First, recall that all sufficiently large prime numbers p are admissible in the sense of [GG2]. Thus, under such iterations of H , the orbit γ stays isolated, and

$$\text{HF}(\gamma^p) = \text{HF}(\gamma)$$

by the persistence of local Floer homology theorem in [GG2]. In particular, in our case, $\text{HF}(\gamma^p) \neq 0$ since $\text{HF}(\gamma) \neq 0$. From now on, we work with primes that are large enough and hence admissible, which we order as $p_1 < p_2 < \dots$. In what follows, p or p_i always denotes a prime from this sequence.

Proceeding with the proof, assume that p_i is a sufficiently large prime number such that φ_H has no simple p_i -periodic orbit in the homology class $p_i[\gamma]$. (If such a prime does not exist then we are done.) Our goal is to show that in this case φ_H must have a simple p_{i+1} -periodic orbit in the homology class $p_i[\gamma]$.

Let $\gamma_1 = \gamma, \gamma_2, \dots, \gamma_r$ be the set of one-periodic orbits of φ_H such that $[\gamma_l] = [\gamma]$ for $l = 1, \dots, r$. Due to the above assumption, any p_i -periodic orbit of φ_H with homology class $p_i[\gamma]$ is the p_i th iteration of some one-periodic orbit β of φ_H . On the other hand, clearly $[\beta^{p_i}] = p_i[\gamma]$ if and only if $[\beta] = [\gamma]$. Thus β must be one of the orbits γ_l . In particular,

$$\mathcal{S}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i}) \subset p_i \bigcup_{l=1}^r \mathcal{S}(H, \llbracket \gamma_l \rrbracket),$$

where, as in Section 2, $\llbracket \gamma \rrbracket$ is the free homotopy class of γ and $\llbracket \gamma \rrbracket^{p_i} := \llbracket \gamma^{p_i} \rrbracket$.

Assume without loss of generality that $\mathcal{A}_H(\gamma) = 0$, which we can ensure by adding a constant to H ; hence $\mathcal{A}_{H^{\natural p}}(\gamma^p) = 0$ for all iterations p by (2.1). Following closely [Gü], pick $a > 0$ outside $\bigcup_{l=1}^r \mathcal{S}(H, \llbracket \gamma_l \rrbracket)$ such that 0 is the only action value in $(-a, a)$. Therefore, 0 is the only point of $\mathcal{S}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i})$ in the interval $(-p_i a, p_i a)$, and hence

$$\text{HF}^{(-p_i a, p_i a)}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i}) = \text{HF}(\gamma^{p_i}) \oplus \dots, \quad (3.1)$$

where the dots represent the local Floer homology contributions from the remaining one-periodic orbits γ_l with zero action such that $\llbracket \gamma_l \rrbracket^{p_i} = \llbracket \gamma \rrbracket^{p_i}$.

Let $C = \|H\|$ and choose p_i so large that $p_i a > 6C(p_{i+1} - p_i)$. The latter is guaranteed by the fact that $p_{i+1} - p_i = o(p_i)$; see, e.g., [BHP]. Now, pick $\alpha > 0$ such that $p_i a - 4C(p_{i+1} - p_i) < \alpha < p_i a - 2C(p_{i+1} - p_i)$. Then we have

$$-p_i a < -\alpha < -\alpha + 2C(p_{i+1} - p_i) < 0 < \alpha < \alpha + 2C(p_{i+1} - p_i) < p_i a,$$

and also

$$-p_{i+1}a < -\alpha + C(p_{i+1} - p_i) < 0 < \alpha + C(p_{i+1} - p_i) < p_{i+1}a.$$

Set $\delta_i := C(p_{i+1} - p_i)$ and denote by $(a, b) + \epsilon$ the shifted interval $(a + \epsilon, b + \epsilon)$. For a linear homotopy from $H^{\natural p_i}$ to $H^{\natural p_{i+1}}$, we have the induced map

$$\mathrm{HF}^{(-\alpha, \alpha)}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i}) \rightarrow \mathrm{HF}^{(-\alpha, \alpha) + \delta_i}(H^{\natural p_{i+1}}, \llbracket \gamma \rrbracket^{p_i})$$

in filtered Floer homology. A proof of this fact for the ordinary filtered Floer homology can be found in [Gil]. This is also true in our case since continuation maps preserve homotopy classes of periodic orbits. Likewise, the linear homotopy from $H^{\natural p_{i+1}}$ to $H^{\natural p_i}$ results in another action shift in δ_i . Consider now the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HF}^{(-\alpha, \alpha)}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i}) & & \\ \downarrow & \searrow \cong & \\ \mathrm{HF}^{(-\alpha, \alpha) + \delta_i}(H^{\natural p_{i+1}}, \llbracket \gamma \rrbracket^{p_i}) & \longrightarrow & \mathrm{HF}^{(-\alpha, \alpha) + 2\delta_i}(H^{\natural p_i}, \llbracket \gamma \rrbracket^{p_i}) \end{array}$$

Here the filtered Floer homology groups of $H^{\natural p_i}$ are non-zero by (3.1) and the choice of α . Furthermore, the diagonal map is an isomorphism induced by the natural quotient-inclusion map; see, e.g., [Gil]. Indeed, zero is the only action value in the intervals $(-\alpha, \alpha)$ and $(-\alpha, \alpha) + 2\delta_i$, and the map is an isomorphism by the stability of the filtered Floer homology; see, e.g., [GG2].

We now have a non-zero isomorphism factoring through the filtered homology of $H^{\natural p_{i+1}}$ in the commutative diagram. In particular, this group cannot be zero:

$$\mathrm{HF}^{(-\alpha, \alpha) + \delta_i}(H^{\natural p_{i+1}}, \llbracket \gamma \rrbracket^{p_i}) \neq 0.$$

Thus φ_H must have a p_{i+1} -periodic orbit η in the homotopy class $\llbracket \gamma \rrbracket^{p_i}$, and hence in the homology class $p_i[\gamma]$. What remains to show is that η is necessarily simple, provided that p_{i+1} (i.e., p_i) is large enough. Arguing by contradiction, assume that $\eta = \nu^{p_{i+1}}$ for some one-periodic orbit ν . Then

$$p_i[\gamma] = [\eta] = p_{i+1}[\nu]. \quad (3.2)$$

Let us write $[\gamma] = n\sigma$ and $[\nu] = m\rho$ for some prime homology classes σ and ρ in $H_1(M; \mathbb{Z})/\mathrm{Tor}$, where $n, m \in \mathbb{N}$. Using (3.2), we now see that $\sigma = \rho$ and $np_i = mp_{i+1}$, where n is fixed – it is determined by γ . This is clearly impossible once p_{i+1} is greater than n . □

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